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A NEW METHOD OF APPORTIONMENT OF REPRESENTATIVES*

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I. THE CONSTITUTIONAL REQUIREMENTS

The only provisions of the United States Constitution which bear on the present problem of apportionment are the following: "Representatives shall be apportioned among the several states according to their respective numbers, counting the whole number of persons in each state, excluding Indians not taxed." "The enumeration shall be made within . . . every . . . term of ten years." "The number of representatives shall not exceed one for every 30,000, but each state shall have at least one representative."

Since fractional representation appears to be out of the question, the problem is how to distribute any given total number (N) of representatives among the several states in proportion to their populations "as nearly as may be."

In other words, considering first the case of two states, the problem is how to approximate most closely to the condition expressed by the proportion

$$A : B :: a : b,$$

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* The method here proposed was first presented at a meeting of the American Mathematical Society, December 28, 1920, and an abstract of the technical results will be published in the *Proceedings of the National Academy of Sciences*, probably in April, 1921.

The new method was brought to the attention of Congress through two letters to Chairman Isaac Siegel of the House Committee on the Census. The first (dated January 8, and published in the *New York Times* for January 16, 1921, Section VII) showed that in the apportionment bill for 483 members then pending, the relation between Missouri and Montana could be improved by changing from 16 and 2 to 15 and 3. The second (dated January 17, and published in the *Congressional Record* for January 19, 1921, page 1791) showed that in the revised apportionment bill for 435 members, which was eventually passed by the House but allowed to die in the Senate, the following improvements could be made:

	Old	New		Old	New		Old	New
New York........	43	42	North Carolina..	11	10	Virginia........	10	9
New Mexico......	1	2	Rhode Island...	2	3	Vermont.......	1	2

Also, two letters to Senator Howard Sutherland, chairman of the Senate Committee on the Census, dated January 22 and 29, 1921, urged investigation of the new method, and in February Senator Sutherland referred the matter for advice to the Advisory Committee to the director of the census. The report of this committee is expected to be available before the apportionment bill is taken up afresh in the new session of Congress, which has been called for April 11, 1921.

Further publications on the mathematical aspects of this subject may be expected either in the *Transactions of the American Mathematical Society* or in the *American Mathematical Monthly*.

(The manuscript for this article was received by the editor on April 15, 1921.)

where A, B, are the populations of the two states, and a, b, the number of representatives assigned to each.

When we come to translate this proportion into the form of a numerical equation, we find that this can be done in four ways:

$$A/a = B/b; \text{ or } a/A = b/B; \text{ or } A/B = a/b; \text{ or } B/A = b/a;$$

no one of which has any constitutional claim to be preferred over any of the others. The problem is equivalent, therefore, to the problem of how to make all four of these equations as nearly true as possible. We are thus led to the following:

FUNDAMENTAL PRINCIPLE. *In each of the four equations*

$$A/a = B/b, \quad a/A = b/B, \quad A/B = a/b, \quad B/A = b/a,$$

the quantity on the left and the quantity on the right should be as nearly equal as possible.

This fourfold requirement may be expressed in words as follows:

(I*a*) The *number of people per representative* (that is, the size of a congressional district) in state A, and the number of people per representative in state B, should be as nearly equal as possible. ($A/a = B/b$.)

(I*b*) The *fractional share of a representative per inhabitant* in state A, and the fractional share of a representative per inhabitant in state B, should be as nearly equal as possible. ($a/A = b/B$.)

(I*c*) If state A is, say, *twice as populous* as state B, it should have *twice as many representatives*, as nearly as may be. ($A/B = a/b$.)

(I*d*) If state A is, say, *half as populous* as state B, it should have *half as many representatives*, as nearly as may be. ($B/A = b/a$.)

The validity of this fourfold fundamental principle, as far as it goes, can hardly be called in question, since, as we have seen, it is an inevitable consequence of the plain provisions of the Constitution. To make this principle precise, however, it is necessary to decide what meaning shall be attached to the words "nearly equal," and on this point there is the possibility of a difference of opinion.

II. THE PROPER MEANING OF "NEARLY EQUAL"

There are two measures of inequality in common use: (1) the *absolute* difference, as when we say, the temperatures of two rooms differ by 10 degrees; and (2) the *percentage* difference, as when we say, the wages in two cities differ by 10 per cent.

Since we are dealing here with proportions rather than with absolute numbers, we shall adopt the second of these interpretations.

DEFINITION OF INEQUALITY. *By the "inequality" between two quantities, we shall mean their percentage difference.*

For example, the inequality between 2 and 3 is 50 per cent, while the inequality between 100 and 105 is 5 per cent. Hence 100 and 105 are more "nearly equal" than 2 and 3 are.*

Applying this definition to the four cases mentioned in the fundamental principle, we readily see that any "inequality" that may exist will be precisely the same in all four cases; that is, if the quantities of any one pair are as nearly equal as possible, the quantities of each of the other pairs will also be as nearly equal as possible. In other words, on this interpretation of "nearly equal," the four parts of the fundamental principle are merely four different ways of stating one single requirement. (This would not be true on the other interpretation of "nearly equal"; see Appendix I.)

III. PRECISE FORMULATION OF THE PROBLEM

We are now in position to formulate definitely the conditions or postulates which an apportionment ought, if possible, to satisfy.

For the case of two states, we have

Postulate I. *For two states, A and B, the percentage difference between A/a and B/b (or between a/A and b/B, or between A/B and a/b, or between B/A and b/a, which all come to the same thing) should be as small as possible.*

For the case of three or more states one further postulate is needed.

Postulate II. *In a satisfactory apportionment, there should be no pair of states which is capable of being "improved" by a transfer of representatives within that pair*—the word "improvement" being understood in the sense implied by Postulate I, and the rare case of a "tie" being decided in favor of the larger state.

On the basis of these two simple postulates, the following theorem can now be established:

Theorem I. *For any given values of A, B, C, . . . and N, there will always be one and only one "satisfactory" apportionment in the sense defined by Postulates I and II.* (See "working rule" below.)

The method thus developed may be called the *Method of Equal Proportions.*

IV. THE METHOD OF EQUAL PROPORTIONS: WORKING RULE

A practical working rule for computing the required apportionment in any given case may be stated as follows:

First, assign one representative to each state (here 48 in number).

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* "Percentage difference" is here thought of as the absolute difference divided by the smaller number. For the present purpose it might equally well be thought of as the absolute difference divided by the larger number, or the absolute difference divided by the mean (arithmetic, geometric, or harmonic) between the two numbers.

Next, multiply the population of each state by as many of the numbers

$$1/\sqrt{1\times 2}, \qquad 1/\sqrt{2\times 3}, \qquad 1/\sqrt{3\times 4}, \quad \cdots$$

as may be necessary, and record each result, together with the name of the state, on a small card. (A table of multipliers is given below.) Then arrange these cards according to the magnitude of the numbers recorded upon them, from the largest to the smallest, thus forming a priority list" for the given states. Finally, assign further representatives, beginning with the 49th, to the several states in the order in which the names of the states occur in this priority list, until the required total, N, has been reached.

In practice, the multiplications for any state may stop with the last result which is greater than P/N', where P is the total population of all the states, and N' is a number larger than the largest value of N likely to be required.

The proof that this working rule will always give an apportionment satisfying the conditions of Postulates I and II is a straight-forward piece of algebra which need not be given here.

It will be observed that the "multipliers" are the reciprocals of the geometric means of consecutive integers; hence the method of equal proportions may be called also the *method of the geometric mean.**

TABLE I

TABLE OF MULTIPLIERS

k	$1/\sqrt{(k-1)k}$	Diff.	k	$1/\sqrt{(k-1)k}$	Diff.	k	$1/\sqrt{(k-1)k}$	Diff.
1	....	....	18	.057 1662	.003 4677	35	.028 9886	.000 8655
2	.707 1068	....	19	.054 0732	.003 0930	36	.028 1718	.000 8168
3	.408 2483	.298 8585	20	.051 2989	.002 7743	37	.027 3998	.000 7720
4	.288 6751	.119 5732	21	.048 7950+	.002 5039	38	.026 6690	.000 7308
5	.223 6068	.065 0683	22	.046 5242	.002 2708	39	.025 9762	.000 6928
6	.182 5742	.041 0326	23	.044 4554	.002 0688	40	.025 3185−	.000 6577
7	.154 3033	.028 2709	24	.042 5628	.001 8926	41	.024 6932	.000 6253
8	.133 6306	.020 6727	25	.040 8248	.001 7380	42	.024 0981	.000 5951
9	.117 8511	.015 7795	26	.039 2232	.001 6016	43	.023 5310	.000 5671
10	.105 4093	.012 4418	27	.037 7426	.001 4806	44	.022 9900	.000 5410
11	.095 3463	.010 0630	28	.036 3696	.001 3730	45	.022 4733	.000 5167
12	.087 0388	.008 3075	29	.035 0931	.001 2765	46	.021 9793	.000 4940
13	.080 0641	.006 9747	30	.033 9032	.001 1899	47	.021 5066	.000 4727
14	.074 1249	.005 9392	31	.032 7913	.001 1119	48	.021 0538	.000 4528
15	.069 0066	.005 1183	32	.031 7500+	.001 0413	49	.020 6197	.000 4341
16	.064 5497	.004 4569	33	.030 7729	.000 9771	50	.020 2031	.000 4166
17	.060 6339	.003 9158	34	.029 8541	.000 9188	51	.019 8030	.000 4001

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* This method of equal proportions resembles closely the "method of alternate ratios" proposed by Dr. Joseph A. Hill, chief statistician of the Bureau of the Census, in 1910. (See Appendix II.) In most practical cases Dr. Hill's method will give precisely the same apportionment as the method here proposed; in some cases, however, it can be shown to lead to inconsistent results. To Dr. Hill belongs the credit of being the first to propose and advocate the use of percentage differences in this problem; he also made a partial application of the idea of a priority list based on the geometric mean. In fact, the method of equal proportions, though arrived at through quite independent considerations, is in effect a carrying out to its logical conclusion of the fundamental idea of the method of alternate ratios as proposed by Dr. Hill in 1910. The writer is indebted to Dr. Hill for calling his attention to this problem in the fall of 1920, and for courteous assistance in securing advance figures on population, etc.

V. APPLICATION TO THE 1920 CENSUS

The results of the proposed method of equal proportions as applied to the 1920 census are shown in the accompanying tables. The

TABLE II

TABLES OF APPORTIONMENT (1920) FOR ANY SIZE OF HOUSE FROM 435 TO 484

1910 435	Table for 435			Priority List, showing where each additional member from the 436th to 484th should be assigned	Table for 484		
	State	1920 435	Gain or Loss over 1910		State	1920 484	Gain over 1910
10	Ala.	10		436 N. Y. 43	Ala.	11	1
1	Ariz.	1		437 N. C. 11	Ariz.	1	
7	Ark.	7		438 Va. 10	Ark.	8	1
11	Calif.	14	3	439 La. 8	Calif.	16	5
4	Colo.	4		440 Miss. 8	Colo.	4	
5	Conn.	6	1	441 Tex. 20	Conn.	6	1
1	Del.	1		442 Okla. 9	Del.	1	
4	Fla.	4		443 Pa. 37	Fla.	4	
12	Ga.	12		444 N. Y. 44	Ga.	13	1
2	Idaho	2		445 Mich. 16	Idaho	2	
27	Ill.	27		446 Nebr. 6	Ill.	30	3
13	Ind.	12	—1	447 Kans. 8	Ind.	13	
11*	Iowa	10	—1	448 Calif. 15	Iowa	11	
8	Kans.	7	—1	449 Ill. 28	Kans.	8	
11	Ky.	10	—1	450 Ohio 25	Ky.	11	
8	La.	7	—1	451 Mo. 15	La.	8	
4	Maine	3	—1	452 Ind. 13	Maine	4	
6	Md.	6		453 Ark. 8	Md.	7	1
16	Mass.	16		454 N. J. 14	Mass.	18	2
13	Mich.	15	2	455 Mass. 17	Mich.	17	4
10	Minn.	10		456 N. Y. 45	Minn.	11	1
8	Miss.	7	—1	457 Pa. 38	Miss.	8	
16	Mo.	14	—2	458 Ga. 13	Mo.	16	
2	Mont.	2		459 Ky. 11	Mont.	3	1
6	Nebr.	5	—1	460 Iowa 11	Nebr.	6	
1	Nev.	1		461 Wis. 12	Nev.	1	
2	N. H.	2		462 N. Y. 46	N. H.	2	
12	N. J.	13	1	463 Ill. 29	N. J.	14	2
1*	N. M.	2	1	464 Tex. 21	N. M.	2	1
43	N. Y.	42	—1	465 Minn. 11	N. Y.	47	4
10	N. C.	10		466 Pa. 39	N. C.	12	2
3	N. D.	3		467 Ore. 4	N. D.	3	
22	Ohio	24	2	468 Ohio 26	Ohio	26	4
8	Okla.	8		469 W. Va. 7	Okla.	9	1
3	Ore.	3		470 S. C. 8	Ore.	4	1
36	Pa.	36		471 Ala. 11	Pa.	40	4
3	R. I.	3		472 Md. 7	R. I.	3	
7	S. C.	7		473 N. Y. 47	S. C.	8	1
3	S. D.	3		474 Tenn. 11	S. D.	3	
10	Tenn.	10		475 N. C. 12	Tenn.	11	1
18	Tex.	19	1	476 Mich. 17	Tex.	21	3
2	Utah	2		477 Maine 4	Utah	2	
2	Vt.	2		478 Calif. 16	Vt.	2	
10	Va.	9	—1	479 Mont. 3	Va.	11	1
5	Wash.	6	1	480 Pa. 40	Wash.	6	1
6	W. Va.	6		481 Mass. 18	W. Va.	7	1
11	Wis.	11		482 Va. 11	Wis.	12	1
1	Wyo.	1		483 Ill. 30	Wyo.	1	
				484 Mo. 16			

* If this method had been used in 1910, Iowa would have had 10 and New Mexico 2.

TABLE III

PEOPLE PER REPRESENTATIVE FOR HOUSE OF 435 MEMBERS

1920	No. of Reps.	Congressional District
U. S....	435	241,864
Ala.....	10	234,817
Ariz....	1	309,754
Ark.....	7	250,314
Calif....	14	244,716
Colo....	4	234,790
Conn....	6	230,105
Del.....	1	223,003
Fla.....	4	242,118
Ga.....	12	241,319
Idaho...	2	215,221
Ill......	27	240,196
Ind.....	12	244,199
Iowa....	10	240,402
Kans....	7	252,751
Ky......	10	241,663
La......	7	256,930
Maine ..	3	256,005
Md.....	6	241,610
Mass....	16	240,772
Mich....	15	244,561
Minn....	10	238,566
Miss....	7	255,803
Mo.....	14	243,147
Mont...	2	270,756
Nebr....	5	259,274
Nev.....	1	75,820
N. H....	2	221,542
N. J....	13	242,762
N. M...	2	176,714
N. Y....	42	247,166
N. .C...	10	255,912
N. D....	3	214,915
Ohio....	24	239,975
Okla....	8	253,535
Ore.....	3	261,130
Pa......	36	242,223
R. I....	3	201,466
S. C....	7	240,532
S. D....	3	210,413
Tenn. ..	10	233,789
Tex.....	19	245,433
Utah....	2	224,194
Vt......	2	176,214
Va......	9	256,576
Wash...	6	225,766
W. Va...	6	243,950
Wis.....	11	239,210
Wyo....	1	193,487

rapidity of the method can be judged from the fact that these results can be recomputed (with the aid of Barlow's Tables of square roots and reciprocals and a calculating machine) in less than three hours.

Tables qualities which remain between the several states, as indicated, for example, by the varying size of the congressional district,

TABLE IV
POPULATION DATA, 1920

State	Total population 1920	Indians not taxed	Population basis of apportionment
Ala.	2,348,174	......	2,348,174
Ariz.	334,162	24,408	309,754
Ark.	1,752,204	......	1,752,204
Calif.	3,426,861	830	3,426,031
Colo.	939,629	468	939,161
Conn.	1,380,631	......	1,380,631
Del.	223,003	......	223,003
Fla.	968,470	......	968,470
Ga.	2,895,832	......	2,895,832
Idaho	431,866	1,424	430,442
Ill.	6,485,280	......	6,485,280
Ind.	2,930,390	......	2,930,390
Iowa	2,404,021	......	2,404,021
Kans.	1,769,257	......	1,769,257
Ky.	2,416,630	......	2,416,630
La.	1,798,509	......	1,798,509
Maine	768,014	......	768,014
Md.	1,449,661	......	1,449,661
Mass.	3,852,356	......	3,852,356
Mich.	3,668,412	......	3,668,412
Minn.	2,387,125	1,469	2,385,656
Miss.	1,790,618	......	1,790,618
Mo.	3,404,055	......	3,404,055
Mont.	548,889	7,378	541,511
Nebr.	1,296,372	......	1,296,372
Nev.	77,407	1,587	75,820
N. H.	443,083	......	443,083
N. J.	3,155,900	......	3,155,900
N. M.	360,350	6,922	353,428
N. Y.	10,385,227	4,240	10,380,987
N. C.	2,559,123	......	2,559,123
N. D.	646,872	2,126	644,746
Ohio	5,759,394	......	5,759,394
Okla.	2,028,283	......	2,028,283
Ore.	783,389	......	783,389
Pa.	8,720,017	......	8,720,017
R. I.	604,397	......	604,397
S. C.	1,683,724	......	1,683,724
S. D.	636,547	5,308	631,239
Tenn.	2,337,885	......	2,337,885
Tex.	4,663,228	......	4,663,228
Utah	449,396	1,008	448,388
Vt.	352,428	......	352,428
Va.	2,309,187	......	2,309,187
Wash.	1,356,621	2,025	1,354,596
W. Va.	1,463,701	......	1,463,701
Wis.	2,632,067	762	2,631,305
Wyo.	194,402	915	193,487
Total	105,273,049	60,870	105,212,179
D. C.	437,571	......	....
Total	105,710,620	......	....

TABLE V
DETAILS OF THE COMPUTATION REQUIRED IN THE APPLICATION OF THE PROPOSED METHOD TO THE 1920 APPORTIONMENT

Condensed Priority List for Indices Above 244,500		
1920		
Ariz.	1	......
Del.	1	......
Wyo.	1	......
Nev.	1	......
Mont.	2	382,906
Ore.	3	319,817
Utah	2	317,058
Maine	3	313,540
N. H.	2	313,307
Idaho	2	304,368
Nebr.	5	289,878
Fla.	4	279,573
La.	7	277,516
Miss.	7	276,298
Kans.	7	273,002
Va.	9	272,140
Colo.	4	271,112
Okla.	8	271,041
Ark.	7	270,371
N. C.	10	269,755
W. Va.	6	267,244
Md.	6	264,671
N. D.	3	263,216
S. C.	7	259,804
S. D.	3	257,702
Ind.	12	255,058
Ky.	10	254,735
Calif.	14	253,954
Iowa	10	253,406
Mich.	15	253,144
N. J.	13	252,674
Mo.	14	252,325
Tex.	19	252,156
Conn.	6	252,068
Ga.	12	252,050
Minn.	10	251,470
Wis.	11	250,885
N. Y.	42	250,162
N. M.	2	249,911
Vt.	2	249,204
Mass.	16	248,669
Ala.	10	247,519
Wash.	6	247,314
R. I.	3	246,744
Tenn.	10	246,435
Pa.	36	245,659
Ohio	24	245,136
Ill.	27	244,771

Complete Priority List for Indices Between 244,500 and 219,000			
436	N. Y.	43	244,275
437	N. C.	11	244,003
438	Va.	10	243,410
439	La.	8	240,336
440	Miss.	8	239,281
441	Tex.	20	239,219
442	Okla.	9	239,035
443	Pa.	37	238,927
444	N. Y.	44	238,659
445	Mich.	16	236,795
446	Nebr.	6	236,684
447	Kans.	8	236,427
448	Calif.	15	236,419
449	Ill.	28	235,867
450	Ohio	25	235,126
451	Mo.	15	234,902
452	Ind.	13	234,619
453	Ark.	8	234,148
454	N. J.	14	233,931
455	Mass.	17	233,583
456	N. Y.	45	233,295
457	Pa.	38	232,554
458	Ga.	13	231,852
459	Ky.	11	230,417
460	Iowa	11	229,214
461	Wis.	12	229,026
462	N. Y.	46	228,167
463	Ill.	29	227,589
464	Tex.	21	227,542
465	Minn.	11	227,463
466	Pa.	39	226,513
467	Ore.	4	226,145
468	Ohio	26	225,902
469	W. Va.	7	225,854
470	S. C.	8	224,997
471	Ala.	11	223,890
472	Md.	7	223,688
473	N. Y.	47	223,260
474	Tenn.	11	222,909
475	N. C.	12	222,743
476	Mich.	17	222,430
477	Maine	4	221,707
478	Calif.	16	221,149
479	Mont.	3	221,071
480	Pa.	40	220,778
481	Mass.	18	220,225
482	Va.	11	220,172
483	Ill.	30	219,872
484	Mo.	16	219,731
485	Ariz.	2	219,029

are unavoidable inequalities. Any attempt to reduce the inequality between any two states by transferring a representative from one state to the other will be found to result in increasing that inequality instead of reducing it.

VI. A MEASURE OF TOTAL OR AVERAGE ERROR

The method of equal proportions, as developed above, provides a complete solution of the problem of finding the *best* apportionment for any given values of A, B, C, . . . and N; but it does not provide any measure of comparison between two apportionments which are less good than the best; nor any measure of comparison between two apportionments involving different values of N.

Although such a measure of error is of little or no importance in practice, it may be of interest to note that a satisfactory measure, E, can be supplied as follows:*

Let a be the actual and $\alpha = (A/P)N$ the theoretical number of representatives assigned to a typical state A. Then

$$e = \frac{(a/A)-(\alpha/A)}{\sqrt{(a/A)(\alpha/A)}} = \frac{(A/\alpha)-(A/a)}{\sqrt{(A/\alpha)(A/a)}} = \frac{a-\alpha}{\sqrt{a\alpha}}$$

may be regarded as expressing the *error affecting an individual citizen* of that state. Taking the sum of the squares of all such errors for all the inhabitants of the country, we have as the *total error*

$$T = A\,\frac{(a-\alpha)^2}{a\alpha} + B\,\frac{(b-\beta)^2}{b\beta} + \quad . \; . \; .;$$

whence, dividing by the total population, P, and taking the square root, we have

$$E = \sqrt{T/P}$$

as the *average error* of the whole apportionment. For the purpose of computation, it is convenient to write E in the form:

$$E = \sqrt{T'/N}, \text{ where } T' = (a-\alpha)^2/a + (b-\beta)^2/b + \quad . \; . \; .$$

It can be shown that in any given case the method which makes the value of this total or average error a minimum, is precisely the method of equal proportions.

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* The expression E for the average error is new, and is to be presented to the American Mathematical Society on April 23, 1921. An earlier but less satisfactory expression was proposed by Professor F. W. Owens on February 26, 1921 (see Appendix II).

APPENDIX I

CRITICISM OF FOUR ALTERNATIVE METHODS BASED ON A WRONG INTERPRETATION OF "NEARLY EQUAL"

If the inequality between two quantities were to be measured by their *absolute difference,* instead of by their percentage difference, the four parts of our fundamental principle would lead to four distinct and conflicting methods of apportionment, which may be called Methods I*a*, I*b*, I*c*, I*d*. Since there is no mathematical ground for choosing between I*a* and I*b*, or between I*c* and I*d*, the result would be a hopeless dilemma, which shows conclusively that the "absolute difference" interpretation is not the interpretation that is appropriate to the present problem.

In contrast with these four conflicting methods, the method of equal proportions, which may be called Method I, involves no such dilemma, since it satisfies all four of the requirements of the fundamental principle simultaneously.

The following notes on Methods I*a*, I*b*, I*c*, I*d* may be of interest.

Method I*a* leads to a "working rule" with the following "multipliers":

$$\frac{1+2}{2(1\times2)},\quad \frac{2+3}{2(2\times3)},\quad \frac{3+4}{2(3\times4)},\quad \frac{4+5}{2(4\times5)},\quad \ldots$$

and may therefore be called the *method of the harmonic mean.* This method, whenever it differs from Method I, favors the small states more than Method I does. (See Example 1*a* and Example 1.)

Method I*b*, taken in connection with the requirement that every state shall have at least one representative, leads to a "working rule" with the following "multipliers":

$$2/3,\quad 2/5,\quad 2/7,\quad 2/9,\quad \ldots$$

and may therefore be called the *method of the arithmetic mean.* This method, whenever it differs from Method I, favors the large states more than Method I does. (See Example 1*b* and Example 1.)*

The amount by which the results of Methods I*a* and I*b* may diverge is often large, as illustrated in Examples 2, 2*a*, and 2*b*.

Methods I*c* and I*d* also determine two distinct methods (for which no "working rules" have been devised). It can be shown that I*c* favors the small states even more than I*a* does, while I*d* favors the large states even more than I*b* does. (See Examples 1*c* and 1*d*, and Example 1.)

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* The working rule for Method I*b* was first employed by Professor W. F. Willcox in 1910, as the basis of his "method of major fractions" (see Appendix II). The existence of the correlative Method I*a* was not known at that time.

All four of these conflicting methods should therefore be rejected in favor of Method I, which satisfies all four parts of our fundamental principle simultaneously.

APPENDIX II

CRITICISM OF OTHER METHODS OF APPORTIONMENT*

1. *The Vinton Method of 1850.* In this method, long in use in Congress, the rule of procedure is as follows:

First, compute the exact quota for each state, putting aside for separate consideration those states whose quotas are less than one. Next, assign to each of the remaining states a number of representatives equal to the largest whole number contained in the exact quota for that state, and add the results. Then ascertain, by subtraction, how many additional representatives must be assigned "for fractions." Finally, as-

Example 1a

	Ia	I Ib
A 454	4	5
B 246	3	2
700	7	7

Example 1b

	I Ia	Ib
A 451	4	5
B 249	3	2
700	7	7

Example 1c

	Ic	I, Ia Ib, Id
A 564	5	6
B 136	2	1
700	7	7

Example 1d

	I, Ia Ib, Ic	Id
A 548	5	6
B 152	2	1
700	7	7

Example 1

	Ic	Ia	I	Ib	Id
A 564	5	6	6	6	6
B 548	5	5	5	5	6
C 454	4	4	5	5	5
D 451	4	4	4	5	5
E 249	3	3	3	2	2
F 246	3	3	2	2	2
G 152	2	2	2	2	1
H 136	2	1	1	1	1
2800	28	28	28	28	28

Example 2a

	Ia	I	Ib
A 9420	92	93	96
B 147	2	2	1
C 146	2	2	1
D 144	2	2	1
E 143	2	1	1
10000	100	100	100

Example 2

	Ia	I	Ib
A 9430	92	94	96
B 144	2	2	1
C 143	2	2	1
D 142	2	1	1
E 141	2	1	1
10000	100	100	100

Example 2b

	Ia	I	Ib
A 9440	92	95	96
B 143	2	2	1
C 140	2	1	1
D 139	2	1	1
E 138	2	1	1
10000	100	100	100

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* On the Vinton, Hill, and Willcox methods, see papers by J. A. Hill and W. F. Willcox in *House of Representatives Report No. 12*, pp. 1–108, of the Sixty-second Congress, First Session, April 25, 1911; also the presidential address of W. F. Willcox, read at the annual meeting of the American Economic Association, December, 1915, and published in the *American Economic Review*, vol. 6, no. 1, Supplement, pp. 1–16, March, 1916. Tables for the 1920 apportionment according to the Willcox method are printed in *House of Representatives Report* No. 1173, pp. 1–28, of the Sixty-Sixth Congress, Third Session, January 8, 1921. On the d'Hondt method, see John H. Humphreys, *Proportional Representation*, London, 1911, pp. 178–180 and 188–190. On the Owens theory of the Willcox method, nothing has yet been published (April 1, 1921).

sign these additional representatives to as many states as may be necessary, taking the states, for this purpose, in the order of magnitude of the fractions themselves (that is, the state with the largest fraction is the first to receive an additional representative).

This method sometimes leads to absurd results, as shown in Example 3. Here when the total number of representatives is increased from 100 to 101, the representation of one of the states actually decreases from 11 to 10. This paradox first came to the attention of Congress in the tables prepared in 1881, which gave Alabama 8 members when $N=299$ and only 7 members when $N=300$; it is therefore known as the *Alabama Paradox.* (See Willcox, *loc. cit.*, p. 10.) Any method which is capable of producing such a result as this is clearly unsatisfactory.

2. *The Hill Method of alternate ratios.* This method, proposed by Dr. Joseph A. Hill in 1910, proceeds as in the Vinton method, except that for the purpose of assigning the additional representatives "for fractions" the states are arranged in order of magnitude of the quantity $A/\sqrt{x(x+1)}$, where x is the number already assigned to state A (that is, the largest whole number in the true quota for that state) and $x+1$ is the next larger number.

As already stated in a foregoing footnote, this method is the best of all hitherto known methods, Dr. Hill's researches of 1910 containing the first discussion of the problem which was based on the correct principle of percentage differences. In fact, if Dr. Hill had dropped the arbitrary requirement that the best assignment to any state must be either x or $x+1$, he would have been led immediately to the method of equal proportions. As it stands, his method of alternate ratios might some time produce an Alabama Paradox, as is shown by Example 4.

3. *The Willcox Method of major fractions,* devised by Professor W. F. Willcox of Cornell University in 1910, was used in the 1910 apportionment, and again in the bill which passed the House on January 19, 1921.

In this method, the population of each state is divided by as many of the numbers $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, . . . as may be necessary, and all the results are then arranged in a priority list in order of magnitude. The representatives, beginning with the 49th, are then assigned to the several states in the order in which the names of the states occur in this priority list.

This method (though arrived at by quite different reasoning involving the use of a "variable ratio" or divisor) is equivalent in its results to Method I*b* (the method of the arithmetic mean, taken in connection with the requirement that every state shall have at least one representative), and is therefore open to all the objections that have already been raised against that method. It satisfies part I*b* of our fundamental principle only on the assumption that absolute differences are more significant than percentage differences, which is, to say the least, a questionable assumption; and it fails altogether to satisfy parts I*a*, I*c*, and I*d*, no matter which interpretation of "nearly equal" is adopted. (See Example 1.)

It may be added that the name "method of major fractions" is somewhat misleading. *A method which would assign an additional representative for every major fraction in the true quota would indeed have much to recommend it; but no such method is possible.* The Willcox "major fraction" is not a major fraction of the true quota, but a major fraction of an artificial quota, scaled up or down from the true quota (by means of a sliding divisor) in order to meet the requirements of the particular method of computation.

Example 3. Vinton Method. (Paradox)

	N=100		N=101	
	Quota		*Quota*	
A 453	*45.3*	45	*45.753*	46
B 442	*44.2*	44	*44.642*	45
C 105	*10.5*	11	*10.605*	10
1000	*100.0*	100	*101.000*	101

Example 4. Hill Method. (Paradox)

	N=100		N=101	
	Quota		*Quota*	
A 154,550	*30.91*	30	*31.2191*	31
B 154,500	*30.90*	30	*31.2090*	31
C 154,450	*30.89*	30	*31.1989*	31
D 7,400	*1.48*	2	*1.4948*	2
E 7,350	*1.47*	2	*1.4847*	2
F 7,300	*1.46*	2	*1.4746*	2
G 7,250	*1.45*	2	*1.4645*	1
H 7,200	*1.44*	2	*1.4544*	1
500,000	*100.00*	100	*101.0000*	101

Example 5.

	d'Hondt	I,Ia,Ib,Ic,Id
A 9802	99	98
B 198	1	2
10000	100	100

For example, in the 1920 apportionment for 435 representatives, the "ratio" or divisor used in the Willcox method, in order to produce the desired number of major fractions, is 1 to 242,415; whereas the true ratio, in the sense in which the phrase "1 to 30,000" is used in the Constitution, is 1 to 241,867.

4. *The d'Hondt method,* devised by Professor Victor d'Hondt of the University of Ghent, Belgium, is now widely used in European elections.

In this method, the population of each state (or political party) is divided by as many of the numbers 1, 2, 3, 4, . . . as may be necessary, and all the results are then arranged in a priority list in order of magnitude. The representatives, beginning with the first, are then assigned to the several states in the order in which the names of the states occur in this priority list.

In this plan, some states may receive several representatives before other states receive any, so that in order to meet the United States requirement that every state shall have at least one representative, the rule would have to be modified by changing "beginning with the first" to "beginning with the 49th." With or without this modification, the method favors the larger states more than does any other known method, as illustrated in Example 5.

This method therefore violates all four parts of our fundamental principle, no matter which interpretation of "nearly equal" is adopted.

5. *The Owens method of least error* (so-called), which was proposed by Professor F. W. Owens of Cornell University at a recent meeting of the American Mathematical Society, February 26, 1921, is not a new method of apportionment, but a new mathematical basis for the Willcox method of major fractions.

Professor Owens, having in mind the theory of Least Squares, starts with a selected expression for the "total error" of an apportionment, namely:

$$Q = A[(a/A) - k]^2 + B[(b/B) - k]^2 + C[(c/C) - k]^2 + \quad . \quad . \quad . \quad ,$$

and then shows that the method which minimizes this total error Q is precisely the Willcox method.

Here k denotes the true value of a/A or b/B or c/C, . . . It may be noted, however, that any other value of k, even $k=0$, might be substituted without affecting the final result.

In the opinion of the present writer, there is no sufficient reason for selecting the fraction a/A rather than the fraction A/a, or for selecting the absolute difference rather than the percentage difference, in building up the expression for Q. If one must use the idea of total error (which is not required in the method of equal proportions), a better expression for it would be the value T given in a previous paragraph.

In any case, the failure of the Willcox method to meet most of the requirements of our fundamental principle is not in any way affected by the Owens theory. That fundamental principle is an inevitable consequence of the plain provisions of the Constitution, with nothing read between the lines; and the only method which can satisfy the whole of that principle is the method of equal proportions.